Math 62: 11.3 Logarithmic Functions

Math 72: 9.3 Logarithmic Functions.

Logarithmic Functions

objectives

1) Write an exponential equation as an equivalent logarithmic equation.
2) Write a logarithmic equation as an equivalent exponential equation.
3) Solve logarithmic equations which can be written as exponential equations with a common base
4) Evaluate log expressions
5) Graph logarithmic functions.
6) Identify properties of logs (any valid base b)

   $\log_b(1) = 0$

   $\log_b(b) = 1$

   domain of $f(x) = \log_b(x)$ $\quad x > 0$ or $(0, \infty)$

   range of $f(x) = \log_b(x)$ $\quad$ all reals or $(-\infty, \infty)$

7) Use the fact that the log and exponential functions with the same base are inverses

   $f(f^{-1}(x)) = x \qquad \log_b b^k = k$

   $f^{-1}(f(x)) = x \qquad b^{\log_b k} = k$

8) All of these objectives, applied the common log.

   $\log x$ (abbreviation for $\log_{10} x$)

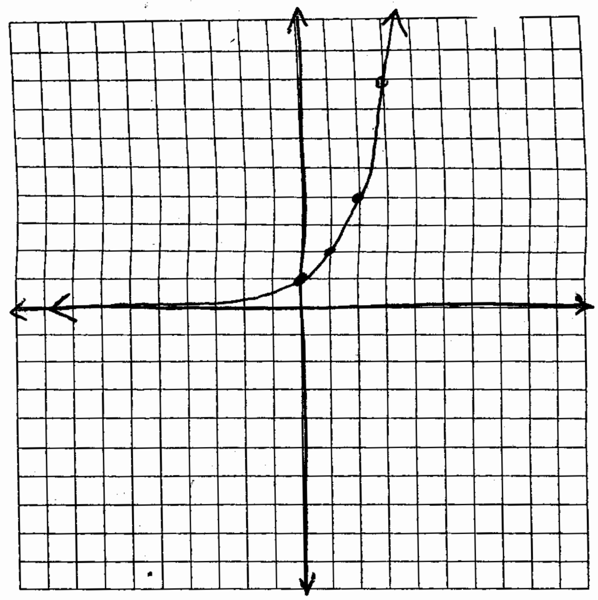
9) All of these objectives, applied to the natural log

   $\ln x$ (abbreviation for $\log_e x$)

Recall: Exponential function

$f(x) = 2^x$

x	$f(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



Passes V.L.T. (it's a function);
Passes H.L.T. (it's one-to-one)
So it's invertible!

x	$f^{-1}(x)$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

But we don't have any algebra to describe $f^{-1}(x)$!
This function transcends (surpasses) algebra, so it's called a <u>transcendental</u> function.

But what is it?

We know that if $f(x) = 2^x$ then

$$f^{-1}\left(\tfrac{1}{8}\right) = -3$$
$$f^{-1}\left(\tfrac{1}{4}\right) = -2$$
$$f^{-1}\left(\tfrac{1}{2}\right) = -1$$
$$f^{-1}(1) = 0$$
$$f^{-1}(2) = 1$$
$$f^{-1}(4) = 2$$
$$f^{-1}(8) = 3$$

The notation mathematicians created for an $f^{-1}(x)$ of this type is called a logarithm.

$$f^{-1}(x) = \log_2(x) \text{ or just } \log_2 x$$

Pronounce: "log-base-2-of-x"

Write the 2 below the log, as a subscript.

The base is crucial.

① If $f(x) = 3^x$, what is $f^{-1}(x)$?

$\boxed{f^{-1}(x) = \log_3 x}$

② If $f(x) = \log_4 x$, what is $f^{-1}(x)$?

$\boxed{f^{-1}(x) = 4^x}$

More terminology: $y = \log_2(x)$

x is the input, also called <u>argument</u>

2 is the <u>base</u>

y is the <u>logarithm</u> or the <u>value of the logarithm</u>

So if (a,c) is a pair on the graph of $f(x) = b^x$ $\quad c = b^a$, then this means (c,a) is a pair on the graph of $f^{-1}(x) = \log_b x$ $\quad a = \log_b c$

$c = b^a$ exponential eq'n	is equivalent to	$a = \log_b c$ logarithmic equation

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Write each exponential equation as an equivalent logarithmic equation.

(3) $2^3 = 8$ $\boxed{\log_2 8 = 3}$

(4) $9^3 = 729$ $\boxed{\log_9 729 = 3}$

(5) $6^{-2} = \frac{1}{36}$ $\boxed{\log_6\left(\frac{1}{36}\right) = -2}$

(6) $\sqrt[3]{5} = 5^{1/3}$ $\boxed{\log_5 \sqrt[3]{5} = \frac{1}{3}}$

(7) $\pi^4 = x$ $\boxed{\log_\pi x = 4}$

Common mistake

Since 2 and 3 are both on the LHS of the exponential, it's tempting to put both 2 and 3 on the LHS of the logarithmic, but that's wrong.

Write each logarithmic equation as an exponential equation.

(8) $\log_5 25 = 2$ $\boxed{5^2 = 25}$

(9) $\log_6 \frac{1}{6} = -1$ $\boxed{6^{-1} = \frac{1}{6}}$

(10) $\log_2 \sqrt{2} = \frac{1}{2}$ $\boxed{2^{1/2} = \sqrt{2}}$

If all the parts are numbers, you should get a true statement.

(11) $\log_7 x = 5$ $\boxed{7^5 = x}$

(12) $\log_{13} y = 4$ $\boxed{13^4 = y}$

Find the value of each logarithmic expression.

(13) $\log_4 16$

step 1: Set expression equal to a variable you choose

$\log_4 16 = Q$

step 2: Write equivalent exponential equation

$4^Q = 16$

<u>step 3</u>: Solve the exponential equation by writing both sides with a common base, then set exponents =.

$4^Q = 4^2$

$Q = 2$

$\boxed{\log_4 16 = 2}$

* Work toward doing these steps in your head *

$\log_4 16$ means 4 raised to <u>what</u> equals 16?

(14) $\log_{10}\left(\frac{1}{10}\right) = Q$

$10^Q = \frac{1}{10}$

$10^Q = 10^{-1}$

$Q = \boxed{-1}$

or 10 raised to the <u>what</u> equals $\frac{1}{10}$?

(15) $\log_9 3 = Q$

$9^Q = 3$

$(3^2)^Q = 3^1$

$2Q = 1$

$Q = \boxed{\frac{1}{2}}$

or $9^{1/2} = 3$

(16) $\log_3 9 = Q$

$3^Q = 9$

$3^Q = 3^2$

$Q = \boxed{2}$

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Solve each equation for the unknown variable.

(17) $\log_4 \frac{1}{4} = x$

Step 1: write equivalent exponential equation.

$4^x = \frac{1}{4}$

$4^x = 4^{-1}$

$\boxed{x = -1}$

(18) $\log_5 x = 3$

step 1: write equivalent exponential equation

$5^3 = x$

$\boxed{x = 125}$

(19) $\log_x 25 = 2$

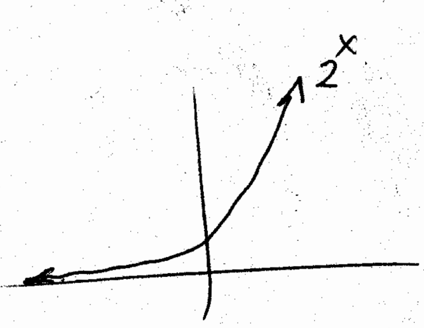
$x^2 = 25$

$x = \pm 5$

However... a base can only be a positive number

so $x = -5$ is extraneous

$\boxed{x = 5}$



(20) $\log_2(-2) = x$

$2^x = (-2)$

has $\boxed{\text{no solution}}$

2 raised to any number is always positive

(21) $\log_3 1 = x$

$3^x = 1$

$3^x = 3^0$

$\boxed{x = 0}$

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(22) $\log_b 1 = x$ solve for x

$b^x = 1$

$b^x = b^0$ for any valid base $b>0$, $b \neq 1$.

$\boxed{x=0}$

(23) $\log_x \frac{1}{7} = \frac{1}{2}$

$x^{1/2} = \frac{1}{7}$

$(x^{1/2})^2 = \left(\frac{1}{7}\right)^2$

$\boxed{x = \frac{1}{49}}$

(24) $\log_4 2^4 = x$

$4^x = 2^4$

$(2^2)^x = 2^4$

$2^{2x} = 2^4$

$2x = 4$

$\boxed{x=2}$

(25) $\log_{3/4} x = 3$

$\left(\frac{3}{4}\right)^3 = x$

$\boxed{x = \frac{27}{64}}$

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(26) If $f(x) = 4^x$ find $f^{-1}(x)$.

$\boxed{f^{-1}(x) = \log_4 x}$

(27) If $g(x) = \log_3 x$, find $g^{-1}(x)$

$\boxed{g^{-1}(x) = 3^x}$

(28) If $f(x) = 5^x$, write $f(f^{-1}(x))$ unsimplified and simplified.

$f(x) = 5^x$

$f^{-1}(x) = \log_5 x$

$f(f^{-1}(x)) = 5^{\log_5 x}$ unsimplified

$f(f^{-1}(x)) = x$ for all inverse functions

so $5^{\log_5 x} = x$

(29) If $f(x) = 5^x$, write $f^{-1}(f(x))$.

$f^{-1}(f(x)) = \boxed{\log_5(5^x) = x}$

Math 70. M-G 9.5 Log Functions, Day 2

Recall:

If $f(x) = b^x$ $\quad$ $f^{-1}(x) = \log_b(x) = \log_b x$

OR if $f(x) = \log_b x$ $\quad$ $f^{-1}(x) = b^x$

logs + exponentials with the same base are inverses of each other

Solve the equations.

① $\log_2 1 = x$

② $\log_{1/3} 1 = x$

③ $\log_{10} 1 = x$

④ $\log_\pi 1 = x$

$x=0$ for all

Property #1
$\log_b 1 = 0$
for any valid base b.
$(b>0,\ b\neq 1)$

Solve the equations:

⑤ $\log_2 2^5 = x$ $\quad x=5$

⑥ $\log_{1/3} (1/3)^{-4} = x$ $\quad x=-4$

⑦ $\log_{10} 10^2 = x$ $\quad x=2$

⑧ $\log_\pi \pi^{-1} = x$ $\quad x=-1$

Property # 2
$\log_b b^x = x$
for any valid base b.
$(b>0,\ b\neq 1)$
Inverse function property

⑨ If $f(x) = 2^x$ and $f^{-1}(x) = \log_2 x$, write the expression for $(f^{-1} \circ f)(x)$.

$= f^{-1}(f(x))$

$= f^{-1}(2^x)$

$= \log_2 2^x = x.$

$(f^{-1} \circ f)(x) = x.$ That's what inverse functions are supposed to do!

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(10) If $f(x) = 2^x$ and $f^{-1}(x) = \log_2 x$, write the expression for $(f \circ f^{-1})(x)$.

$$f(f^{-1}(x))$$
$$= f(\log_2 x)$$
$$= 2^{\log_2 x} = x$$

But if these are really inverses, this is x!

Property #3:

$$b^{\log_b x} = x \quad \text{for any valid base } b$$

Inverse function property

Simplify:

(11) $\log_3 3^2 = 2$

(12) $\log_7 7^{-1} = -1$

(13) $5^{\log_5 3} = 3$

(14) $2^{\log_2 6} = 6$

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Sketch graphs of these logarithmic functions.

* CAUTION * #1	Just plotting points from the usual table of x-values is not enough.
* CAUTION * #2	Just graphing what you see on your GC is not enough.
* CAUTION * #3	Your GC only has $\log_{10}(x)$ and $\log_e(x)$ anyway.

(15) $y=\log_2 x$

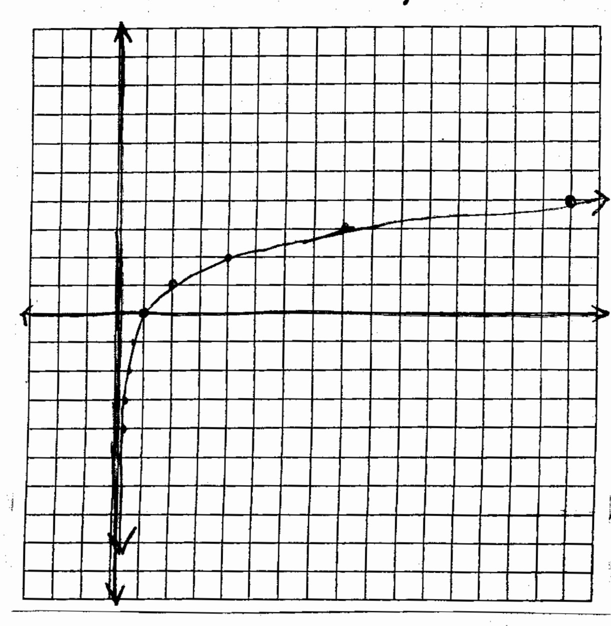
step 1: Find the equivalent exponential function and make a table of values for it.
Clearly label it exponential.

x	2^x
-4	$1/16$
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8
4	16

x	$\log_2 x$
$1/16$	-4
$1/8$	-3
$1/4$	-2
$1/2$	-1
1	0
2	1
4	2
8	3
16	4

step 2: Swap all the ordered pairs (a,b) on exponential graph to get ordered pairs (b,a) on the logarithmic graph. Clearly label it logarithmic.
Cross out the exponential ~~table~~ if necessary

step 3: Plot the swapped points.



step 4: Check that 3 areas are graphed and clear

1) logarithmic growth
2) x-intercept
3) vertical asymptote.

domain: $x>0$ or $(0,\infty)$
range: all reals or $(-\infty,\infty)$

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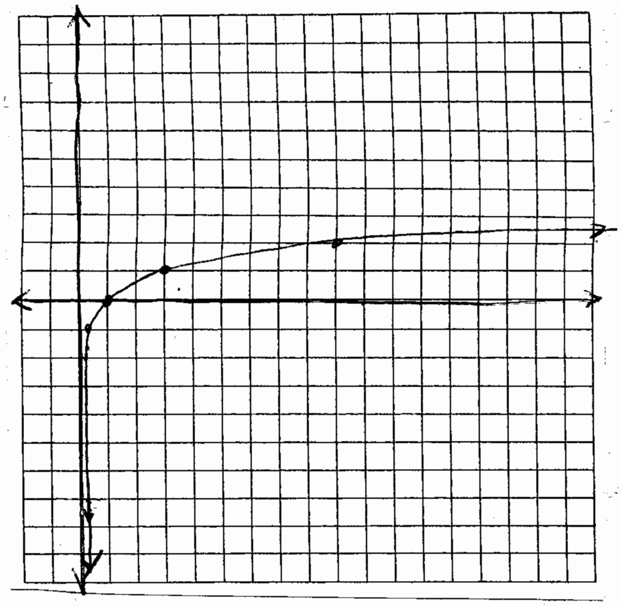
(16) $y = \log_3 x$

x	3^x
-4	$1/81$
-3	$1/27$
-2	$1/9$
-1	$1/3$
0	1
1	3
2	9
3	27
4	81

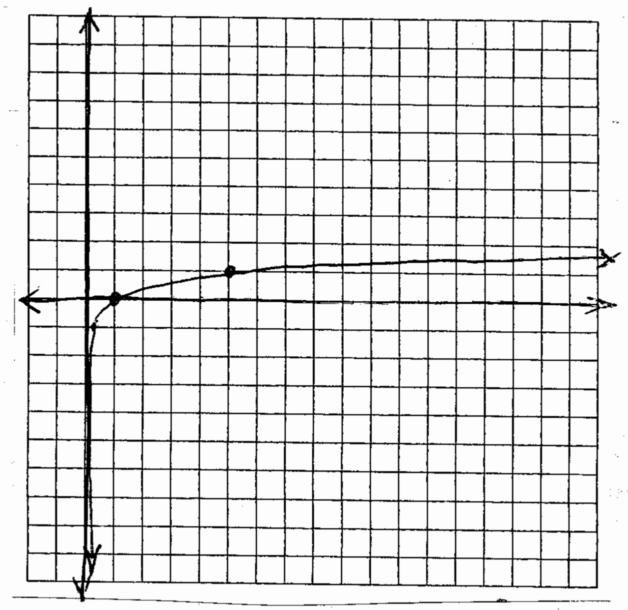
x	$\log_3 x$
$1/81$	-4
$1/27$	-3
$1/9$	-2
$1/3$	-1
1	0
3	1
9	2
27	3
81	4



(17) $y = \log_5 x$

x	5^x
-3	$1/125$
-2	$1/25$
-1	$1/5$
0	1
1	5
2	25
3	125

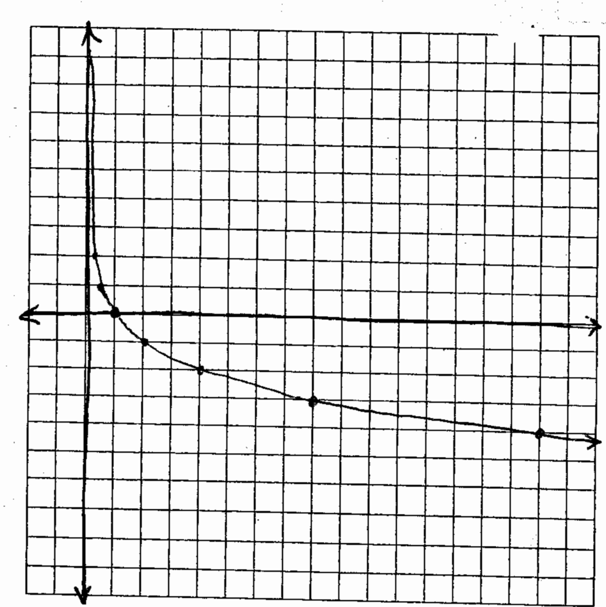
x	$\log_5 x$
$1/125$	-3
$1/25$	-2
$1/5$	-1
1	0
5	1
25	2
125	3



(18) $y = \log_{1/2} x$

x	$(1/2)^x$
-4	$(\frac{1}{2})^{-4} = 2^4 = 16$
-3	8
-2	4
-1	2
0	1
1	$1/2$
2	$1/4$
3	$1/8$
4	$1/16$

x	$\log_{1/2} x$
16	-4
8	-3
4	-2
2	-1
1	0
$\frac{1}{2}$	1
$1/4$	2
$1/8$	3
$\frac{1}{16}$	4



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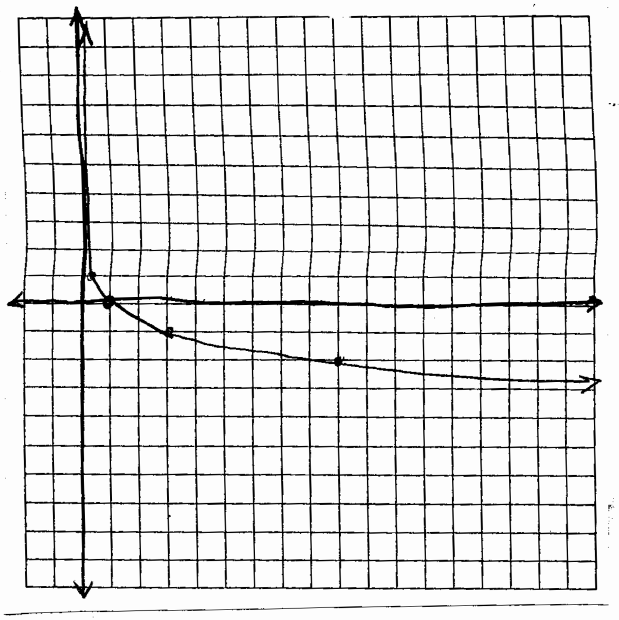
(19) $y = \log_{1/3} x$

x	$(1/3)^x$
-4	$(1/3)^{-4} = (3)^4 = 81$
-3	$(1/3)^{-3} = 3^3 = 27$
-2	9
-1	3
0	1
1	$1/3$
2	$1/9$
3	$1/27$
4	$1/81$

x	$\log_{1/3} x$
81	-4
27	-3
9	-2
3	-1
1	0
$1/3$	1
$1/9$	2
$1/27$	3
$1/81$	4



(20) $y = -\log_2 x$

step 1: Make table for 2^x

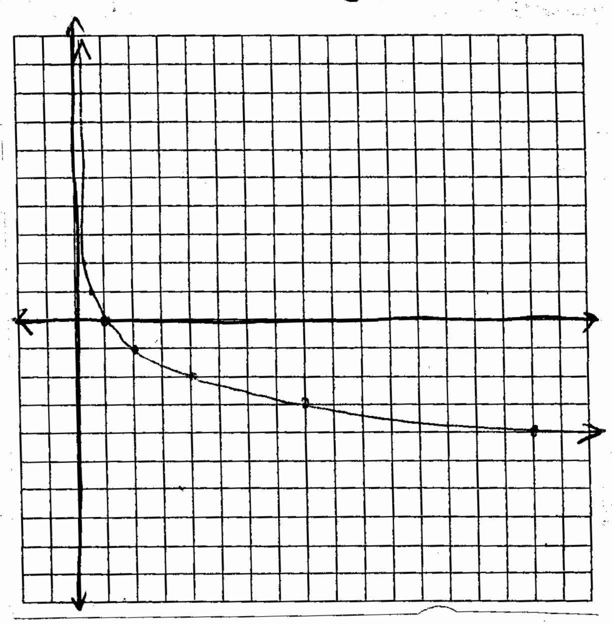
step 2: Swap to get table for $\log_2 x$

step 3: Keep all x-coords of $\log_2 x$ table, but make all y-coords opposite

x	2^x
-4	$1/16$
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8
4	16

x	$\log_2 x$
$1/16$	-4
$1/8$	-3
$1/4$	-2
$1/2$	-1
1	0
2	1
4	2
8	3
16	4

x	$-\log_2 x$
$1/16$	$+4$
$1/8$	$+3$
$1/4$	$+2$
$1/2$	$+1$
1	0
2	-1
4	-2
8	-3
16	-4



Notation: $\log x$ means $\log_{10} x$ and is called the common log. This log is the $\boxed{\text{LOG}}$ button on GC
abbreviation
meaning

If there's no base written, there is still a base! You assume base 10.

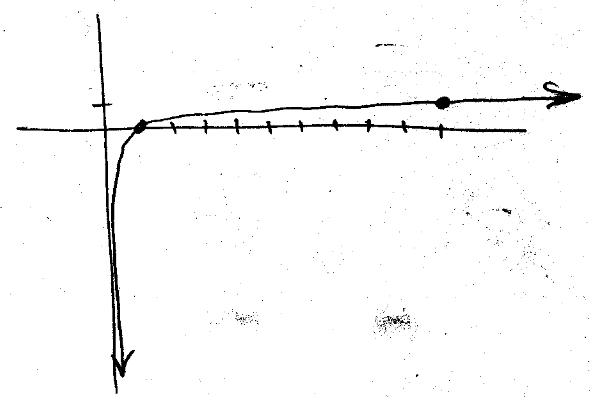
① Approximate $\log 7$ to nearest ten-thousandth.
exact.

$= 0.84509$

$\approx \boxed{0.8451}$ ← approximate answer

This means $10^1 = 10 < 7$
$10^2 = 100 > 7$
$10^{.85} \approx 7$

② Graph $y_1 = \log x$ on GC



Notation: $\ln x$ means $\log_e x$ and is called the natural log. This is the $\boxed{\text{LN}}$ button on GC.

Above the $\boxed{\text{LN}}$, the second function is e^x.

e is an irrational number.

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⑦ $\ln 7$

$\boxed{\text{LN}}$ 7 $\boxed{)}$ $\boxed{\text{ENTER}}$

$= 1.94591$

$\approx \boxed{1.9459}$ ← approximate

$\ln 7$ ← exact

This means
$e^1 = e \approx 2.7$
$e^2 \approx 7.4$

bigger base, smaller exponent
⇓
$10^{\sim .85} = 7$

then $e^{\sim 1.9} \approx 7$
⇑
smaller base, bigger exponent

⑧ $\ln 100$

$= 4.60517$

$\approx \boxed{4.6052}$ ← approximate

$\ln 100$ ← exact

$\ln 7$ and $\ln 100$ are also irrational numbers w/ decimals that do not terminate or repeat.

- if we need an exact answer, we write a simplified log expression

Math 70

Recall Inverse functions un-do each other.

$$f(f^{-1}(x)) = x$$
$$f^{-1}(f(x)) = x$$

Recall: $\left.\begin{array}{l} f(x) = b^x \\ f^{-1}(x) = \log_b x \end{array}\right\}$ are inverse functions. for any valid base b.

If base is 10:

$\left.\begin{array}{l} f(x) = 10^x \\ f^{-1}(x) = \log x \end{array}\right\}$ inverse functions

If base is e:

$\left.\begin{array}{l} f(x) = e^x \\ f^{-1}(x) = \ln x \end{array}\right\}$ inverse functions

If we compose $f(f^{-1}(x)) = \boxed{10^{\log x} = x}$

or $\boxed{e^{\ln x} = x}$

If we compose $f^{-1}(f(x)) = \boxed{\log 10^x = x}$

or $\boxed{\ln e^x = x}$

inverse properties for base 10 and base e.

In particular, this means that

$\boxed{\log 10 = 1}$ because $10^1 = 10$

$\boxed{\ln e = 1}$ because $e^1 = e$

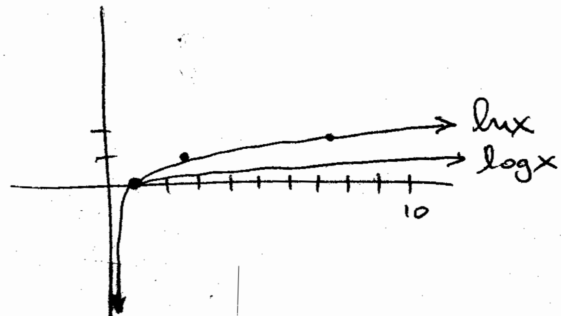
⑨ $\ln 0$

$= \boxed{\text{undefined}}$

⑩ $\ln 1$

$= \boxed{0}$

⑪ Graph $y = \ln x$. How is it different from graph of $\log x$?



The base e is between 2 and 3 so its y values are higher than base 10 log.

Solve by brain, not GC:

(12) $\log 10 = x$

$10^x = 10$

$\boxed{x=1}$

(13) $\log 10^5 = x$

$\boxed{x=5}$

inverse property

✓ (14) $\log \frac{1}{10} = x$

$\boxed{x=-1}$

✓ (15) $\log \sqrt{10} = x$

$\boxed{x=\frac{1}{2}}$

✓ (16) $10^{\log 6} = x$

$\boxed{x=6}$

inverse property

(17) $\log x = -3$

equivalent exponential

$10^{-3} = x$

$\boxed{x=.001}$

✓ (18) $\log x = \frac{1}{3}$

$10^{1/3} = x$

$\boxed{x=\sqrt[3]{10}}$

Solve by brain, not GC.

(19) $\ln e^3 = x$

$e^x = e^3$

$\boxed{x = 3}$

✓ (20) $\ln \sqrt[5]{e} = x$

$e^x = \sqrt[5]{e}$

$\boxed{x = 1/5}$

✓ (21) $e^{\ln 2} = x$ inverse property

$\boxed{x = 2}$

✓ (22) $\log_2 x = 4$

$2^4 = x$

$\boxed{x = 16}$

✓ (23) $\log_x 2 = 3$

$x^3 = 2$

$\boxed{x = \sqrt[3]{2}}$

Solve each equation a) exactly
b) approximately, to 4 decimal places

✓ (24) $\log x = 1.2$

a) $\boxed{10^{1.2} = x}$ ← exact

$10^{1.2}$ is an irrational number: its decimal does not repeat or terminate. We write $10^{1.2}$ if we need an exact answer.

b) $x = 15.84893$

$\boxed{x \approx 15.8489}$ ← approximate

(25) $\ln 3x = 5$

a) $e^5 = 3x$

$\boxed{x = \frac{e^5}{3}}$ exact

b) $\frac{e^5}{3} \approx 49.47105$

$\boxed{49.4711}$ approx

(26) $\ln 5x = 8$

a) $e^8 = 5x$

$\boxed{x = \frac{e^8}{5}}$ exact

b) $\frac{e^8}{5} \approx 596.19159$

$= \boxed{596.1916}$ approx.